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Monotonicity and *orthant-monotonicity of certain maximum norms

Boris Lavrič*

Department of Mathematics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia

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Abstract

Let $\mathbf{K}$ be the field of real or complex numbers. A characterization of all inner product norms p_1 and p_2 on $\mathbf{K}^n$ for which the norm $x \mapsto \max\{p_1(x), p_2(x)\}$ on $\mathbf{K}^n$ is monotonic or *orthant-monotonic is given.

Keywords: Monotonic norms; Orthant monotonic norms; Inner product norms

1. Introduction

Let $\mathbf{K}^n$ be the n-dimensional real or complex vector space of column vectors $x = (x_1, \ldots, x_n)^{\mathrm{T}}$, and let $\mathbf{K}^{n,n}$ be the space of all $n \times n$ matrices with entries in $\mathbf{K}$. A subspace of $\mathbf{K}^n$ generated by a subset of the standard basis $\{e_1, \ldots, e_n\}$ will be called a *coordinate subspace*. For each $C \in \mathbf{K}^{n,n}$ and each nonempty index set $\kappa \subseteq \{1, \ldots, n\}$ we denote by $C[\kappa]$ the principal submatrix of C that corresponds to κ. If C is a positive definite matrix, the functional $p_C : x \mapsto (x^*Cx)^{1/2}$ (x^* is the conjugate transpose of x) is an inner product norm on $\mathbf{K}^n$. As is well known, each norm on $\mathbf{K}^n$ generated by an inner product is of the form p_C for some positive definite matrix $C \in \mathbf{K}^{n,n}$.

A norm p on $\mathbf{K}^n$ is called *monotonic* if $|x| \leqslant |y|$ (componentwise) implies $p(x) \leqslant p(y)$; *absolute* if $p(x) = p(|x|)$ for all $x \in \mathbf{K}^n$; and **orthant-monotonic* if $p(Dx) \leqslant p(x)$ for all $x \in \mathbf{K}^n$ and all diagonal matrices $D = \operatorname{diag}(d_1, \ldots, d_n) \in \mathbf{K}^{n,n}$ such that

* Tel.: +386-61-1766676; fax: +386-61-217281.
E-mail address: lavric@fmf.uni-lj.si (B. Lavrič).

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$d_j \in \{0, 1\}$ for all j. Monotonic norms were introduced in [1] and have been extensively studied. It is well known that monotonicity and absoluteness are equivalent, and easy to see that a norm p is absolute if and only if $p(Dx) \leqslant p(x)$ for all diagonal matrices $D = \mathrm{diag}(d_1, \dots, d_n) \in \mathbf{K}^{n,n}$ such that $|d_j| = 1$ for all j. Monotonicity implies *orthant-monotonicity, which coincides for $\mathbf{K} = \mathbf{R}$ with orthant-monotonicity introduced in [3]. A list of characterizations of monotonic and *orthant-monotonic norms is contained in [4,8]. A large class of *orthant-monotonic norms is given in [7], where the monotonicity properties of some composite norms is discussed.

Let p_1, p_2 be norms on $\mathbf{K}^n$. If p_1 and p_2 are monotonic (resp. *orthant-monotonic), then also the norm $\max\{p_1, p_2\}$ is monotonic (resp. *orthant-monotonic). The converse fails even in case when p_1 and p_2 are inner product norms. In this paper we characterize all inner product norms p_1, p_2 for which the norm $p = \max\{p_1, p_2\}$ is monotonic or *orthant-monotonic. A similar characterization of *orthant-monotonicity of a norm of the form $p = g(p_1, p_2)$ with a differentiable g satisfying some other conditions is given in [5].

Our proofs are based on the characterization [5, Theorem 1] of *orthant-monotonicity of a norm p by its subdifferential ∂p:

*A norm p on $\mathbf{K}^n$ is *orthant-monotonic if and only if for each coordinate subspace W of $\mathbf{K}^n$ and for each $x \in W$ the set $\partial p(x) \cap W$ is nonempty.*

Recall that the *subdifferential* of a norm p at $x \in \mathbf{K}^n$ is the set

$$\partial p(x) := \{v \in \mathbf{K}^n : p(x + y) - p(x) \geqslant \mathrm{Re}(v^* y) \text{ for all } y \in \mathbf{K}^n\}.$$

If the subdifferential $\partial p(x)$ is a one-point set $\{v\}$, we shall write $\partial p(x) = v$. In this case

$$\lim_{y \to 0} \frac{p(x + y) - p(x) - \mathrm{Re}(v^* y)}{p(y)} = 0,$$

i.e., v is the $\mathbf{R}$-differential of p at x. For the proof see for example [6], and use the standard identification of $\mathbf{C}^n$ with $\mathbf{R}^{2n}$ as it is explained in [3]. Every inner product norm on $\mathbf{K}^n$ is $\mathbf{R}$-differentiable at each nonzero $x \in \mathbf{K}^n$. More precisely, for each positive definite matrix $C \in \mathbf{K}^{n,n}$ we have

$$\partial p_C(x) = \frac{Cx}{p_C(x)}, \quad x \in \mathbf{K}^n \setminus \{0\}. \tag{1}$$

We shall need also the following result on the subdifferential (see [2, Proposition 2.3.12]). If $p_1, \dots, p_m$ are norms on $\mathbf{K}^n$, the norm $p = \max\{p_1, \dots, p_n\}$ has the subdifferential

$$\partial p(x) = \mathrm{co}\left(\bigcup_{i \in I(x)} \partial p_i(x)\right), \tag{2}$$

where $I(x) = \{i \in \{1, \dots, m\} : p_i(x) = p(x)\}$.

2. *Orthant-monotonicity

From now on let p_A, p_B be given inner product norms on $\mathbf{K}^n$ defined by positive definite matrices $A = [a_{ij}] \in \mathbf{K}^{n,n}$ and $B = [b_{ij}] \in \mathbf{K}^{n,n}$, let

$$\tau(x) = p_B(x)p_A(x)^{-1}, \quad x \in \mathbf{K}^n \setminus \{0\},$$

and let p be the composite norm $p = \max\{p_A, p_B\}$. Combining (1) and (2) we get the subdifferential

$$\partial p(x) = \begin{cases} p_A(x)^{-1}Ax & \text{if } \tau(x) < 1, \\ p_B(x)^{-1}Bx & \text{if } \tau(x) > 1, \\ p(x)^{-1}\mathrm{co}\{Ax, Bx\} & \text{if } \tau(x) = 1. \end{cases}$$

The results of this note are based on the following characterization of *orthant-monotonicity of the norm p.

Proposition 1. *The norm $p = \max\{p_A, p_B\}$ is *orthant-monotonic if and only if for each coordinate subspace W of $\mathbf{K}^n$*

(a) *$Ax \in W$ for each nonzero $x \in W$ satisfying $\tau(x) < 1$;*
(b) *$Bx \in W$ for each nonzero $x \in W$ satisfying $\tau(x) > 1$;*
(c) *$\mathrm{co}\{Ax, Bx\} \cap W \neq \emptyset$ for each nonzero $x \in W$ satisfying $\tau(x) = 1$.*

Proof. Combine the formula for $\partial p(x)$ with [5, Theorem 1]. $\square$

For the main characterizations we shall need some auxiliary results.

Lemma 2. *Let $p = \max\{p_A, p_B\}$ be *orthant-monotonic, and let W be a coordinate subspace of $\mathbf{K}^n$. Then:*

(a) *If there exists a nonzero $x_0 \in W$ such that $\tau(x_0) < 1$, then $AW \subseteq W$ and $Ae \in \mathbf{K}e$ for each standard basis vector e satisfying $e \notin W$.*
(b) *If there exists a nonzero $x_0 \in W$ such that $\tau(x_0) > 1$, then $BW \subseteq W$ and $Be \in \mathbf{K}e$ for each standard basis vector e satisfying $e \notin W$.*

Proof. (a) Suppose first that $\tau(x_0) < 1$ for some nonzero $x_0 \in W$. Since τ is continuous, $\tau(x) < 1$ for all x in some neighborhood U of x_0 satisfying $0 \notin U$. Proposition 1 ensures that $Ax \in W$ for all $x \in U \cap W$. Since $U \cap W$ is an open subset of W, this implies that $AW \subseteq W$. Take any standard basis vector e such that $e \notin W$, and consider the coordinate subspace $W' = W \oplus \mathbf{K}e$. Since $x_0 \in W'$ and $\tau(x_0) < 1$, the preceding argument shows that $AW' \subseteq W'$, and therefore $Ae \in W'$. Since in addition $A = A^*$, we get $A(W^\perp) \subseteq (W^\perp)$, and consequently $Ae \in W^\perp$. It follows that $Ae \in W' \cap W^\perp = \mathbf{K}e$ as claimed.

The proof of part (b) is similar. $\square$

Corollary 3. *If* $p = \max\{p_A, p_B\}$ *is* **orthant-monotonic*, *then*

(a) $\min\{\tau(e_k) : 1 \leqslant k \leqslant n\} < 1$ *implies that* A *is diagonal*;
(b) $\max\{\tau(e_k) : 1 \leqslant k \leqslant n\} > 1$ *implies that* B *is diagonal*.

Proof. (a) Choose j such that $\tau(e_j) = \min\{\tau(e_k) : 1 \leqslant k \leqslant n\}$, consider the coordinate subspace $W_j = \mathbf{K}e_j$, and use Lemma 2.

The proof of part (b) is similar. □

Lemma 4. *If* $p = \max\{p_A, p_B\}$ *is* **orthant-monotonic*, *there exists a* $\nu \in [0, 1]$ *such that* $\nu A + (1 - \nu)B$ *is diagonal*.

Proof. If A is diagonal take $\nu = 1$, and if B is diagonal take $\nu = 0$. Suppose now that A, B are nondiagonal, and note that Corollary 3 ensures that $\tau(e_k) = 1$ for $k = 1, \ldots, n$. It follows from Proposition 1 that for each $k \in \{1, \ldots, n\}$ there exists a $\nu_k \in [0, 1]$ such that

$$\nu_k A e_k + (1 - \nu_k) B e_k \in \mathbf{K}e_k. \tag{3}$$

Put $N = \{\nu_k : 1 \leqslant k \leqslant n\}$. Furthermore, if $\nu \in N$, put

$$J(\nu) = \{k : \nu_k = \nu\}, \quad A_\nu = A[J(\nu)], \quad B_\nu = B[J(\nu)].$$

It is clear that the family $\{J(\nu) : \nu \in N\}$ is a partition of $\{1, \ldots, n\}$, and it follows from (3) that $\nu A_\nu + (1 - \nu)B_\nu$ is diagonal for every $\nu \in N$.

If $\nu_i \neq \nu_j$, $i, j \in \{1, \ldots, n\}$, then by (3) $\nu_i a_{ji} + (1 - \nu_i) b_{ji} = 0$, and therefore $\nu_i a_{ij} + (1 - \nu_i) b_{ij} = \nu_i \overline{a}_{ji} + (1 - \nu_i) \overline{b}_{ji} = 0$. Moreover, (3) implies $\nu_j a_{ij} + (1 - \nu_j) b_{ij} = 0$, hence $(\nu_i - \nu_j)(a_{ij} - b_{ij}) = 0$, and consequently $a_{ij} = b_{ij} = 0$. It follows that for each $\nu \in N$ the coordinate subspace $W(\nu)$ generated by $\{e_k : k \in J(\nu)\}$ is invariant for A and for B.

Now let $\nu, \xi \in N$, and suppose $0 \leqslant \nu < \xi \leqslant 1$. Assume that in each pair of matrices A_ν, B_ν and A_ξ, B_ξ, at least one is nondiagonal.

If there exists a nonzero $x \in W(\xi)$ such that $\tau(x) < 1$, then A_ν is diagonal by Lemma 2. Since $\nu < 1$ and $\nu A_\nu + (1 - \nu)B_\nu$ is diagonal, B_ν is diagonal as well. This contradicts the assumption, and hence $\tau(x) \geqslant 1$ for all nonzero $x \in W(\xi)$. Suppose that $\nu > 0$. If there exists a nonzero $x \in W(\xi)$ such that $\tau(x) > 1$, then B_ν is diagonal by Lemma 2. It follows that A_ν is diagonal as well. This contradicts the assumption, and hence $\tau(x) \leqslant 1$ for all nonzero $x \in W(\xi)$. Thus $\tau(x) = 1$ for all nonzero $x \in W(\xi)$. It folows that B_ξ and A_ξ are equal and hence diagonal. This contradicts the assumption, and therefore $\nu = 0$. Interchanging the roles of $W(\nu)$ and $W(\xi)$, we get in a similar way $\xi = 1$. Thus, $\nu = 0$ and $\xi = 1$.

Since A_1 is diagonal, B_1 is not diagonal, and hence $Be_k \notin \mathbf{K}e_k$ for some $e_k \in W(1)$. Therefore Lemma 2 ensures that $\tau(x) \leqslant 1$ for each nonzero $x \in W(0)$. This implies that $A_0 - B_0$ is positive semidefinite. Since $\tau(e_k) = 1$ for each $k \in J(0)$, we have $\mathrm{tr}(A_0 - B_0) = 0$, and therefore $A_0 = B_0$. We know that B_0 is diagonal, hence A_0 is diagonal as well, but this contradicts the assumption.

It follows now that $N = \{\nu\}$ for some $\nu \in [0, 1]$, thus (3) completes the proof. □

Theorem 5. *Let* $A, B \in \mathbf{K}^{n,n}$ *be positive definite, and let*

$$\iota \equiv \{j \in \{1, \ldots, n\} : a_{ij} \neq 0 \ \text{or} \ b_{ij} \neq 0 \ \text{for some} \ i \neq j\}.$$

The norm $p = \max\{p_A, p_B\}$ *is* **orthant-monotonic if and only if one of the following conditions is satisfied*:

(a) *A and B are diagonal*;
(b) *A is diagonal,* $(A - B)[\kappa]$ *is positive semidefinite for each* $\kappa = \iota \setminus \{j\}$ *with* $j \in \iota$, *and* $a_{ii} \geqslant b_{ii}$ *for each* $i \in \{1, \ldots, n\} \setminus \iota$.
(c) *B is diagonal,* $(B - A)[\kappa]$ *is positive semidefinite for each* $\kappa = \iota \setminus \{j\}$ *with* $j \in \iota$, *and* $a_{ii} \leqslant b_{ii}$ *for each* $i \in \{1, \ldots, n\} \setminus \iota$.
(d) *A and B are of the form*

$$A = D + E, \quad B = D - tE$$

with D diagonal, $t > 0$, *and*

$$E = \lambda E_{rs} + \overline{\lambda} E_{sr}, \quad \lambda \in \mathbf{K} \setminus \{0\}, \ \ r \neq s,$$

where $E_{rs}, E_{sr} \in \mathbf{K}^{n,n}$ *are elementary matrices*.

Proof. A coordinate subspace generated by $\{e_k : k \in \kappa\}$, $\kappa \subseteq \{1, \ldots, n\}$, will be denoted by W_κ, where we adopt the convention $W_\emptyset = \{0\}$.

Suppose that p is *orthant-monotonic. Observe that ι is a minimal subset of $\{1, \ldots, n\}$ satisfying $Ae \in \mathbf{K}e$ and $Be \in \mathbf{K}e$ for all standard basis vectors $e \notin W_\iota$, and that ι is empty if and only if (a) holds.

Assume now that ι is nonempty. Note that $A = A^*$ and $B = B^*$ implies $|\iota| \geqslant 2$. Take any $\kappa \subseteq \{1, \ldots, n\}$ such that $\iota \setminus \kappa \neq \emptyset$. By Lemma 4 there exists a $\nu \in [0, 1]$ such that $\nu A + (1 - \nu)B$ is diagonal. If $\tau(x) > 1$ for some nonzero $x \in W_\kappa$, then by Lemma 2 $Be_j \in \mathbf{K}e_j$ for each $j \in \iota \setminus \kappa$. If in addition $0 < \nu \leqslant 1$, then $Ae_j \in \mathbf{K}e_j$ for each $j \in \iota \setminus \kappa$ as well. Since this contradicts the definition of ι, $\tau(x) \leqslant 1$ for every nonzero $x \in W_\kappa$. It follows that in this case $(A - B)[\kappa]$ is positive semidefinite. If $0 \leqslant \nu < 1$ we obtain similarly that $(B - A)[\kappa]$ is positive semidefinite. Therefore, $\nu = 1$ gives (b), $\nu = 0$ gives (a), and $0 < \nu < 1$ implies that $A[\kappa] = B[\kappa]$ for each $\kappa \subseteq \{1, \ldots, n\}$ such that $\iota \setminus \kappa \neq \emptyset$. Since $\nu A + (1 - \nu)B$ is diagonal, $A[\kappa] = B[\kappa]$ is diagonal as well. This ensures that ι has two elements, $\iota = \{r, s\}$, hence in this case (d) is satisfied with $t = \nu(1 - \nu)^{-1}$.

To prove the converse suppose first that A and B are diagonal. Then every coordinate subspace W of $\mathbf{K}^n$ is invariant for A and for B, hence p is *orthant-monotonic by Proposition 1.

Take now a coordinate subspace $W = W_\kappa$, $\emptyset \neq \kappa \subseteq \{1, \ldots, n\}$. If $\iota \subseteq \kappa$, then W is invariant for A, B, hence the subspace W satisfies the conditions of Proposition 1. If $\iota \setminus \kappa \neq \emptyset$, take any $j \in \iota \setminus \kappa$. Suppose (b) is satisfied. Then it follows easily that $(A - B)[\kappa]$ is positive semidefinite, and consequently $\tau(x) \leqslant 1$ for each nonzero

$x \in W$. Since A is diagonal and hence $AW \subseteq W$, W satisfies the conditions of Proposition 1. We can see in a similar way that also in the case (c) the subspace W satisfies the same conditions. Suppose now (d) holds. Then $\iota = \{r, s\}$. If $r \notin \kappa$, then $A[\kappa] = B[\kappa]$ are diagonal and $\tau(x) = 1$ for each nonzero $x \in W$. Since $\nu A + (1-\nu)B$ is diagonal for $\nu = t(1+t)^{-1}$, the subspace W is invariant for $\nu A + (1-\nu)B$, and therefore $\mathrm{co}\{Ax, Bx\} \cap W \neq \emptyset$ for each $x \in W$. Thus, Proposition 1 ensures that in all cases p is *orthant-monotonic. $\square$

It seems that a description of all positive definite matrices $A_1, \ldots, A_m \in \mathbf{K}^n$ for which the norm $p = \max\{p_{A_1}, \ldots, p_{A_m}\}$ is *orthant-monotonic is much harder for $m > 2$. Nevertheless, some introductory results of the section can be smoothly generalized on more than two norms. For example, we can see easily (combining (1), (2) and [5, Theorem 1]) that the norm $p = \max\{p_{A_1}, \ldots, p_{A_m}\}$ is *orthant-monotonic if and only if for each coordinate subspace W of $\mathbf{K}^n$ and for each nonzero $x \in W$

$$\mathrm{co}\{A_i x : i \in I(x)\} \cap W \neq \emptyset.$$

In the final section we shall apply Theorem 5 to get a description of all inner product norms p_1, p_2 for which the norm $p = \max\{p_1, p_2\}$ is monotonic.

3. Monotonicity

To obtain a characterization of positive definite matrices A and B for which the norm $p = \max\{p_A, p_B\}$ is monotonic, we need the following result.

Lemma 6. *Let $p = \max\{p_A, p_B\}$ be monotonic and let A be diagonal. If B is not diagonal, then $A - B$ is positive semidefinite and hence $p = p_A$.*

Proof. If $A - B$ is not positive semidefinite, $p_A(x_0) < p_B(x_0)$ for some $x_0 \in \mathbf{K}^n$. Continuity of the norms ensures that there exists a neighborhood U of x_0 such that $p_A(x) < p_B(x)$ and hence $p(x) = p_B(x)$ for all $x \in U$. Take any $D = \mathrm{diag}(d_1, \ldots, d_n) \in \mathbf{K}^{n,n}$ such that $|d_j| = 1$ for all j. Since A is diagonal, p_A is monotonic, thus $p_A(Dx) = p_A(x)$ and consequently

$$p(Dx) = \max\{p_A(Dx), p_B(Dx)\} = \max\{p_A(x), p_B(Dx)\}$$

for all $x \in \mathbf{K}^n$. Monotonicity of p implies $p(x) = p(Dx)$, hence for all $x \in U$ we have

$$p_B(x) = p(x) = p(Dx) = \max\{p_A(x), p_B(Dx)\},$$

and therefore $p_B(x) = p_B(Dx)$. Thus

$$x^*(B - D^*BD)x = 0 \quad \text{for all } x \in U.$$

It is a routine to verify that this implies $B = D^*BD$. Now, for each different i, $j \in \{1, \dots, n\}$ take a D such that $d_i = 1$, $d_j = -1$, and observe that $e_i^*Be_j = e_i^*D^*BDe_j = -e_i^*Be_j$. It follows that B is diagonal. □

Theorem 7. *The norm* $p = \max\{p_A, p_B\}$ *is monotonic if and only if one of the following conditions is satisfied*:

(a) *A and B are diagonal*;
(b) *A is diagonal and $A - B$ is positive semidefinite*.
(c) *B is diagonal and $B - A$ is positive semidefinite*.
(d) $\mathbf{K} = \mathbf{R}$ *and A, B are of the form*

$$A = D + E, \quad B = D - E$$

with D diagonal, and

$$E = \lambda(E_{rs} + E_{sr}), \quad \lambda \in \mathbf{R} \setminus \{0\}, \ \ r \neq s,$$

where $E_{rs}, E_{sr} \in \mathbf{R}^{n,n}$ are elementary matrices.

Proof. Suppose that p is monotonic. If A is diagonal and B is nondiagonal or vice versa, Lemma 6 shows that (b) or (c) is satisfied. If A and B are both nondiagonal, then the fact that p is *orthant-monotonic implies that the condition (d) of Theorem 5 is satisfied. For each $\alpha \in \mathbf{K}$ put $x(\alpha) = \alpha e_r + e_s$. Then

$$x(\alpha)^*Ex(\alpha) = 2\mathrm{Re}(\overline{\alpha}\lambda),$$
$$x(\alpha)^*Dx(\alpha) = |\alpha|^2 d_r + d_s,$$

and consequently

$$p(x(\alpha))^2 = |\alpha|^2 d_r + d_s + 2\max\{\mathrm{Re}(\overline{\alpha}\lambda), -t\,\mathrm{Re}(\overline{\alpha}\lambda)\}.$$

Since p is absolute and $|x(\alpha)| = x(|\alpha|)$, we have $p(x(\overline{\zeta}\lambda)) = p(x(|\zeta|\lambda))$ for each $\zeta \in \mathbf{K}$. It follows easily that

$$\max\{\mathrm{Re}\zeta, -t\,\mathrm{Re}\zeta\} = |\zeta| \quad \text{for each } \zeta \in \mathbf{K}.$$

This is impossible in the case $\mathbf{K} = \mathbf{C}$, and implies $t = 1$ in the case $\mathbf{K} = \mathbf{R}$.

To prove the converse suppose first that A and B are diagonal. Then p_A and p_B are monotonic, hence p is monotonic as well. Suppose now that (b) is satisfied. Then A is diagonal and $p = p_A$, hence p is monotonic. Analogously (c) implies that p is monotonic. If (d) holds with $D = \mathrm{diag}(d_1, \dots, d_n)$, an easy calculation shows that

$$p(x)^2 = \sum_{j=1}^{n} d_j |x_j|^2 + 2|\lambda x_r x_s|,$$

hence $p(x) = p(|x|)$ for all $x \in \mathbf{K}^n$. □

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